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Nonlinear instability and break-up of separated flow

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Unsteady effects in a separated recirculating eddy beneath a subsonic or supersonic mainstream are considered, with emphasis on nonlinear properties. The eddy is slender and predominantly inviscid, its length being comparable with, or less than, the chord of an airfoil, for instance. It is found, from the study of a family of integro-differential equations, that the planar eddy can break up nonlinearly within a finite time, causing an eruption into the main stream and setting up a subsequent Euler stage in the unsteady motion. Comparisons with recent experiments and further applications of the theory are discussed.

1. Introduction

Both small- and large-scale separating flows are well known to be very prone to instability at sufficiently high Reynolds numbers, according to experimental evidence and to the classical linear stability theory associated with infinitesimal disturbances. Such non-parallel basic flows tend to be susceptible to linear instabilities of the Tollmien–Schlichting, Görtler, Rayleigh and Kelvin–Helmholtz types. The current theoretical contribution is aimed at extending the understanding of the instability of these flows to encompass nonlinear features, as distinct from the well-established existence of linear instability. The nonlinear behaviour of imposed or free unsteady disturbances is essential to the process of transition to turbulence in developing boundary layers, separating flows and other basic non-parallel motions. In practice, it is found that a separating laminar flow can break down abruptly in spatial terms with a sudden turbulent reattachment occurring just beyond the separation point, due to transition and the enhanced entrainment of the resulting turbulent motion, and/or with an increase in the lateral scale of the strongly disturbed, unsteady, flow, among other significant features. See, for example, the works of Tani (1964), Gaster (1966), Mueller & Batill (1980), Van Dyke (1982), Dovgal, Kozlov & Simonov (1987), Kozlov (1987), Mezaris *et al.* (1988) and references therein.

Recent theoretical studies of some nonlinear disturbances in separating flow are presented by Smith & Burggraf (1985) and Smith (1985*a*) (again see references therein), while certain linear disturbances properties were addressed recently by Goldstein (1984), Stewart & Smith (1987) and in fact many previous workers (e.g. see Drazin & Reid 1981, p. 229) have used various approximations for the linear instability regime. Of some relevance here is the prediction

$$x^*_{\mathrm{crit}} = x^*_{\mathrm{sep}} + l\left(\frac{\Omega^* l}{3b^2 u^*_1}\right)^{\frac{1}{3}} \lambda_1^{-\frac{7}{4}} Re_1^{-\frac{11}{24}}, \tag{1.1a}$$

(Smith 1985*a*) for the position $x^* = x^*_{\text{crit}}$ at which substantial inviscid instability first arises in a separating incompressible boundary layer for a linear disturbance of fixed frequency Ω^*, with the separation sited at $x^* = x^*_{\text{sep}}$ on an airfoil, say. In (1.1*a*) Re_1 is a global Reynolds number, based on the local free-stream velocity u^*_1 and the airfoil chord, l, and is assumed large, while λ_1 is the $O(1)$ skin-friction factor of the attached boundary layer just ahead of separation, $b \approx 0.44$ and Ω^* lies in the range

$$Re_1^{\frac{1}{4}} \frac{u^*_1}{l} \ll \Omega^* \ll Re_1^{\frac{1}{2}} \frac{u^*_1}{l}. \tag{1.1b}$$

The prediction (1.1*a*) is from a far-downstream analysis of the triple-deck structure describing the laminar separation, and indeed we observe that if the driving frequency Ω^* is reduced to the Tollmien–Schlichting order $Re_1^{\frac{1}{4}} u^*_1/l$ then the critical position in (1.1*a*) moves upstream to within $O(l\,Re_1^{-\frac{3}{8}})$ of separation, i.e. within that triple-deck, yielding then a non-parallel Tollmien–Schlichting instability of the local separating flow. Conversely, if the frequency Ω^* is raised to the order $Re_1^{\frac{1}{2}} u^*_1/l$ then the critical position is moved downstream, to $O(l\,Re_1^{-\frac{7}{24}})$ beyond separation, at which stage the departure distance of the detached boundary layer from the airfoil surface becomes comparable with the detached boundary-layer thickness. This happens because locally the departure distance $\propto l\,Re_1^{-\frac{1}{16}}(x^*/l)^{\frac{3}{2}}$ in laminar separation, whereas the above boundary-layer thickness stays fixed at $O(l\,Re_1^{-\frac{1}{2}})$. At that stage the detailed basic-flow properties of the detached boundary layer/free shear layer and the separated flow underneath it count equally in the linear stability characteristics, which are then more of the Rayleigh kind.

There are two main reasons for our interest in the prediction (1.1). One is that it agrees well qualitatively with the experiments of Mezaris *et al.* (1988), and to some extent quantitatively as well (as their paper shows) bearing in mind the comments following (1.1*b*), and that encourages further study, as here, of the stability characteristics of separating flow at large Reynolds numbers. The second is that, combined with the above comments, and with account taken of the increased departure of the free shear layer further downstream, the framework underlying (1.1) points to an investigation of nonlinear disturbance properties in a separated eddy of longer scale, comparable with the airfoil chord, where the viscous free shear layer appears relatively thin. This sets the scene for the present study of nonlinear disturbances on a more global scale than for (1.1).

The model used here has an unsteady nonlinear eddy motion, typically of length $O(l)$ or less, but slender, of width $O(h)$ say, beneath a subsonic/incompressible or supersonic stream, as described in §2. The existence of the nonlinear breakdowns proposed in §§4, 5 means that such a separated eddy motion, initially in a slowly evolving state of either zero or uniform vorticity, can erupt locally, within a finite scaled time, if a non-infinestimal disturbance perturbs the separated flow. In practice such a disturbance might well emanate from free-stream unsteadiness, for example, or from surface roughness or a vibrating ribbon upstream, while the eddy present may be long or short. The assumption of effectively zero initial flow in the eddy or, more precisely, of initial velocities less than $O(h^{\frac{1}{2}})$ there, would tend to be associated more with a long eddy since the analysis is localized, so that the separation then appears 'open', whereas the uniform-vorticity condition would seem more appropriate perhaps to a short or 'closed' eddy. The local eruption can arise in subsonic or supersonic flow also, with or without finite vorticity, and indeed it can apply in principle to thin eddies with fairly general distributions of vorticity because the breakdown analysis is localized.

Further discussion is given in §6, including extra applications and extensions of the theory as well as comments on the experiments.

2. The nonlinear equations for the inviscid eddy

A slender two-dimensional airfoil of thickness-to-chord ratio h is symmetrically placed in a uniform stream U_∞. The fluid is taken to be inviscid and a free streamline leaves the body at some point ahead of the trailing-edge and forms an eddy to the rear of the airfoil. The velocity components in the direction of, and perpendicular to, the line of symmetry of the flow are $U_\infty u, U_\infty v$, and the corresponding coordinates are lx, ly where l is the chord length and the origin is at the leading edge; the pressure is $\rho_\infty U_\infty^2 p$, where ρ_∞ is the constant density at infinity, and tl/U_∞ is the time. The equation $y = S(x,t)$ of the eddy boundary is one of the unknowns.

On the assumption that the thickness ratio $h \ll 1$, and that the eddy width is also $O(h)$, the flow may be considered in two regions. Outside the airfoil and eddy combination the flow is quasi-steady on a long timescale $t = O(h^{-\frac{1}{2}})$ and is a linear perturbation of the free stream. Thus p is $O(h)$ and the value of $p(x,0,t)$ is, for subsonic flow, given by slender-airfoil theory as

$$p(x,0,t) = -\frac{1}{\pi}\int_0^{x_0}\frac{f'(\eta)}{x-\eta}\mathrm{d}\eta - \frac{1}{\pi}\int_{x_0}^{\infty}\frac{S_\eta(\eta,t)}{x-\eta}\mathrm{d}\eta, \tag{2.1}$$

where $y = f(x)$ is the equation of the airfoil and x_0 is the point of departure of the streamline from the body. We shall not be concerned here with its value which could, in principle, be determined by a triple-deck smooth separation condition or fixed at a corner in the airfoil geometry. If the flow is supersonic (2.1) is to be replaced by the simpler condition

$$p(x,0,t) = f'(x) \quad \text{or } S_x(x,t), \tag{2.2}$$

according as $x \lessgtr x_0$.

In the interior of the eddy the flow is nonlinear and is assumed to be slow moving with $u = O(h^{\frac{1}{2}}), v = O(h^{\frac{3}{2}})$ and $p = O(h)$, in a region in which $x = O(1)$ but $y = O(h)$ since the eddy width is comparable with the thickness of the airfoil. For viscosity to be unimportant it is necessary that $Rh^{\frac{5}{2}} \gg 1$ [see however §6], R being a representative Reynolds number. With this condition and $h \ll 1$ the governing equations in the eddy are

$$u_x + v_y = 0, \quad u_t + uu_x + vu_y = -p_x, \quad 0 = -p_y, \tag{2.3a–c}$$

whence, since the pressure is independent of y, p is given by $p(x,0,t)$ in (2.1) or (2.2). Solutions of the system (2.3) which holds for $x > x_0$ have, as material surfaces of the fluid, $y = S(x,t)$ above, and the airfoil $y = f(x)$ below.

Equation (2.3b) is equivalent to $\mathrm{D}(u_y)/\mathrm{D}t = 0$ and a simple example, which nevertheless contains many features of interest, is obtained by assuming an initially zero distribution of vorticity in the eddy, so that u remains independent of y for all x, y, t. In this case $u = u(x,t), v = -yu_x + (fu)_x$, equation (2.3$b$) reduces to

$$u_t + uu_x = -p_x, \tag{2.4}$$

and the kinematic condition that $v = S_t + uS_x$ at the eddy boundary $y = S(x,t)$ becomes, when y is set equal to S in the expression for v given above,

$$S_t + [u(S-f)]_x = 0, \tag{2.5}$$

with $f \equiv 0$ for values of x beyond the trailing edge.

The system of equations to be solved for u, S, p is thus (2.4), (2.5) together with either (2.1) or (2.2). If instead of a zero vorticity we taken $u_y = G$ for constant negative vorticity G, so that $u = Gy + q(x,t)$, then (2.4) and (2.5) are replaced by

$$q_t + qq_x + G(fq + \tfrac{1}{2}Gf^2)_x = -p_x, \tag{2.6}$$

$$S_t + [q(S-f) + \tfrac{1}{2}G(S^2 - f^2)]_x = 0 \tag{2.7}$$

respectively. The case $G = 0$ corresponds to a limit problem of very weak vorticity in the separated region.

An indication of the possible behavior of the solutions of (2.4), (2.5) subject to (2.1), or (2.2) with $x > x_0$, can be obtained by seeking local short-wave linear perturbations to a solution with $S = S_0, f = f_0$, both constants. On replacing $\partial/\partial t$ by $-\mathrm{i}\Omega$ and $\partial/\partial x$ by $\mathrm{i}\alpha$, where $|\Omega|, |\alpha|$ are both large, we find that

$$\Omega^2 = -(S_0 - f_0)\,\alpha^2|\alpha|, \tag{2.8}$$

if the flow is subsonic, i.e. if (2.1) is used, and

$$\Omega^2 = (S_0 - f_0)\,\mathrm{i}\alpha^3, \tag{2.9}$$

if (2.2) is used for supersonic flow. Thus in either case $\operatorname{Im}\Omega \to \infty$ as $\alpha \to \infty$, with the flow then being linearly unstable. This is indicative of the solution of an appropriate initial-value problem for the linearized system becoming singular in a finite time. There is thus the prospect of such a phenomenon occurring also in the solution of the nonlinear system, and it is to the possible form of such breakdown that we now address ourselves.

3. The similarity equations of the breakdown region

To illustrate the breakdown phenomenon we suppose that it occurs at time $t = t_s - 0$ in the neighbourhood of an unknown station $x = x_s > x_0$. For simplicity we set $f \equiv 0$† and write

$$x - x_s = (t_s - t)^n\,\tilde{x} \quad (n > 0), \tag{3.1}$$

and

$$(S, u, p) = [(t_s - t)^{3n-2}\,\tilde{S}(\tilde{x}), \quad (t_s - t)^{n-1}\,\tilde{u}(\tilde{x}), \quad (t_s - t)^{2n-2}\,\tilde{p}(\tilde{x})] \tag{3.2}$$

in (2.4), (2.5) so that they become

$$(2-3n)\,\tilde{S} + n\tilde{x}\tilde{S}' + (\tilde{u}\tilde{S})' = 0, \tag{3.3a}$$

$$(1-n)\,\tilde{u} + n\tilde{x}\tilde{u}' + \tilde{u}\tilde{u}' = -\tilde{p}', \tag{3.3b}$$

with

$$\tilde{p} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\tilde{S}'(\tilde{\eta})}{\tilde{x} - \tilde{\eta}}\,\mathrm{d}\tilde{\eta}, \tag{3.4}$$

if the mainstream is subsonic, and

$$\tilde{p} = \tilde{S}', \tag{3.5}$$

† This is without loss of generality when $n < \frac{2}{3}$, since then $S \gg 1$ and $f(= O(1))$ is negligible.

if it is supersonic. If G in (2.6), (2.7) is non-zero then, as $t \to t_s - 0$, the term in GSS_x in (2.7) dominates if $n < \frac{1}{2}$, is unimportant at leading order if $n > \frac{1}{2}$, but if $n = \frac{1}{2}$ the equation becomes

$$\tfrac{1}{2}\tilde{S} + \tfrac{1}{2}\tilde{x}\tilde{S}' + (\tilde{u}\tilde{S})' + G\tilde{S}\tilde{S}' = 0, \tag{3.6}$$

where, for convenience and comparison with (3.3a), we have written $q = (t_s - t)^{-\frac{1}{2}}\tilde{u}(\tilde{x})$. Since $f \equiv 0$, equation (2.6) then transforms into (3.3b).

The assumption that breakdown is confined to the neighbourhood of $x = x_s$ means that, if $n < \frac{2}{3}$, $\tilde{S}$, and if $n < 1$, both $\tilde{u}$ and $\tilde{p}$, must tend to zero as $|\tilde{x}| \to \infty$. This is clear from the transformation (3.1). If $n = \frac{2}{3}$ it is acceptable for $\tilde{S}$ to tend to non-zero limits, although in all cases $\tilde{S}$ must remain non-negative over $(-\infty, \infty)$ since it represents the scaled eddy width.

4. Supersonic eddy-breakdown solutions

In this section we examine solutions of (3.3) together with (3.5) when $\frac{1}{2} \leqslant n \leqslant \frac{2}{3}$. For $|\tilde{x}| \gg 1$ we require $\tilde{S}, \tilde{u}$ to tend to zero (or be bounded when $n = \frac{2}{3}$) with $\tilde{S} = O(|\tilde{x}|^{(3n-2)/n})$ and $\tilde{u} = O(|\tilde{x}|^{(n-1)/n})$. We first note that the system is invariant under the transformation

$$(\tilde{S}, \tilde{u}, \tilde{p}, \tilde{x}) \to (A^3\tilde{S}, A\tilde{u}, A^2\tilde{p}, A\tilde{x}), \tag{4.1}$$

for any constant A, but is not reversible up- and down-stream since negative A implies negative $\tilde{S}$ which is not permissible. Secondly, for all n an exact solution is $\tilde{S} = \frac{1}{24}\tilde{x}^3$, $\tilde{u} = -\frac{1}{2}\tilde{x}$ which, although in itself not of direct use, does demonstrate the asymptotic form of the solutions which must be excluded. That no acceptable solutions exist for $n < \frac{1}{2}$ may be seen on integrating (3.3a) in the form

$$[(n\tilde{x} + \tilde{u})\tilde{S}]_{-\infty}^{\infty} + 2(1-2n)\int_{-\infty}^{\infty} \tilde{S}\,\mathrm{d}\tilde{x} = 0; \tag{4.2}$$

the value of the integrated term is zero, and that of the integral, which converges in this situation, is positive.

If $n = \frac{1}{2}$ an analytic solution of the system is possible, and if $n = \frac{2}{3}$ a linearized solution may be found. We first consider $n = \frac{1}{2}$ as it yields phenomena that are also present at larger values of n. It is also, as demonstrated in §6, most likely to correspond to an eddy breakdown of a separated flow as described here, one reason being that, as can be seen from (3.1), (3.2), the lowest acceptable value of n leads to the strongest singularity. We later consider $n = 0.6$ as a representative value between the limits of $\frac{1}{2}$ and $\frac{2}{3}$.

4.1. *The value* $n = \frac{1}{2}$

When $n = \frac{1}{2}$ equations (3.3), (3.5) integrate to

$$(\tfrac{1}{2}\tilde{x} + \tilde{u})\tilde{S} = C, \quad \tfrac{1}{2}(\tilde{x} + \tilde{u})\,\tilde{u} + \tilde{S}' = D, \tag{4.3a, b}$$

where C, D are constants, and we see immediately that (4.3a) is unacceptable with the same non-zero C for all $\tilde{x}$ since $\tilde{u} \to 0$ as $|\tilde{x}| \to \infty$ and such C cannot be both positive and negative. Here we seek a solution in which $C = 0$, and discontinuities in $\tilde{u}$ and $\tilde{S}'$ are permitted. The adjustment regions at these discontinuities are then accommodated by appeal to the full equations for the eddy, namely (2.4), (2.5). To satisfy (4.3a) we assume that $\tilde{S} = 0$ for $\tilde{x} < \tilde{x}_-$ and for $\tilde{x} > \tilde{x}_+$, and that $\tilde{u} = -\frac{1}{2}\tilde{x}$ for

$\tilde{x}_- < \tilde{x} < \tilde{x}_+$. Then (4.3$b$) is satisfied if $\tilde{S}$ is a cubic in $\tilde{x}$ when $\tilde{x}_- < \tilde{x} < \tilde{x}_+$, with zeroes at $\tilde{x}_-, \tilde{x}_+$, and

$$\tilde{u} = -\tfrac{1}{2}\tilde{x} + \tfrac{1}{2}(\tilde{x}^2 + 8D_+)^{\frac{1}{2}} \quad \text{for } \tilde{x} > \tilde{x}_+, \tag{4.4a}$$

$$\tilde{u} = -\tfrac{1}{2}\tilde{x} - \tfrac{1}{2}(\tilde{x}^2 + 8D_-)^{\frac{1}{2}} \quad \text{for } \tilde{x} < \tilde{x}_-, \tag{4.4b}$$

where the roots have been chosen so that $|\tilde{u}| \to 0$ as $|\tilde{x}| \to \infty$, and we shall allow D_+ to differ from D_- if necessary. By trial it is found that the adjustment regions, to be discussed below, can accommodate only the situation in which the cubic for $\tilde{S}$ has a double zero at $\tilde{x} = \tilde{x}_+ > 0$ and a single zero at $\tilde{x} = \tilde{x}_- < 0$, with the result that, since the cubic for $\tilde{S}$ has no term in $\tilde{x}^2, \tilde{x}_- = -2\tilde{x}_+$ and

$$\tilde{S} = \tfrac{1}{24}(\tilde{x} + 2\tilde{x}_+)(\tilde{x} - \tilde{x}_+)^2. \tag{4.5}$$

Thus the eddy boundary has collapsed so that it has a cusped trailing edge and a sharp leading edge. At the cusped downstream end the equation for the adjustment region requires $\tilde{u}$ to be continuous which means that $D_+ = -\tfrac{1}{8}\tilde{x}_+^2$, and at the sharp upstream edge the appropriate adjustment region equation determines the discontinuity in $\tilde{u}$ which, as it will emerge, leads to $D_- = -\tfrac{1}{8}\tilde{x}_+^2$ also.

When the constant vorticity G in (3.6) is non-zero (4.3a) is replaced by

$$(\tfrac{1}{2}\tilde{x} + \tilde{u} + \tfrac{1}{2}G\tilde{S})\tilde{S} = C, \tag{4.6}$$

and upon again choosing $C = 0$, and $\tilde{S} \equiv 0$ for $\tilde{x} > \tilde{x}_+$ and $\tilde{x} < \tilde{x}_-$, it follows that the equation to be satisfied by $\tilde{S}$ in $\tilde{x}_- < \tilde{x} < \tilde{x}_+$ is

$$\tilde{S}' = \tfrac{1}{8}(\tilde{x}^2 - G^2\tilde{S}^2 - \tilde{x}_+^2). \tag{4.7}$$

As in the case $G = 0$, $\tilde{S}$ must have a double zero at $\tilde{x}_+ (> 0)$ since the adjustment region equations will be found to be independent of the value of G at leading order. Equation (4.7) does not have an acceptable solution, i.e. one with a second zero for $\tilde{x} < \tilde{x}_+$, for all values of $\tilde{x}_+^2 G$, however. That this is the relevant parameter may be seen by defining

$$\tilde{x} = \tilde{x}_+ \tilde{X}, \quad \tilde{S}(\tilde{x}) = \tilde{x}_+ \tilde{R}(\tilde{X})/|G| \tag{4.8}$$

so that (4.7) becomes

$$\tilde{R}' = A(\tilde{X}^2 - \tilde{R}^2 - 1), \quad A = \tfrac{1}{8}\tilde{x}_+^2 |G|, \tag{4.9}$$

which is to be solved with $\tilde{R}(1) = 0$ for those values of A for which $\tilde{R}$ has a second zero at some $\tilde{X} < 1$. When $A \ll 1$ we obtain $R = \tfrac{1}{3}A(\tilde{X}+2)(\tilde{X}-1)^2$ so that in this limit we again have the solution (4.5).

As A increases the position of the second zero of $\tilde{R}$ becomes increasingly negative and tends to minus infinity as $A \to A_c$ which has been found numerically to lie between 0.59 and 0.60. For $A > A_c$, $\tilde{R}$ tends to infinity at a finite value of $\tilde{X}$. The limit solution with $A \approx A_c$ may be described as follows. As $A \to A_c - 0$ the maximum value, $\tilde{R}_{\max}$ say, of $\tilde{R}$ increases and, if we scale the solution so that the maximum value of $\tilde{S}$ is unity, it follows from (4.8) that $\tilde{x}_+ \tilde{R}_{\max}/|G| = 1$ with $\tilde{R}_{\max} \gg 1$. Thus, from the definition of A in (4.9),

$$|G| = (\tilde{R}_{\max})^{\frac{2}{3}}(8A_c)^{\frac{1}{3}} \gg 1, \tag{4.10}$$

so that this gives the limiting solution for large values of the vorticity $|G|$. Once this is established it is simpler to describe its properties in terms of the variables $\tilde{S}, \tilde{x}$. When $0 < -\tilde{x} = O(1)$ we have, from (4.7), that

$$\tilde{S} \approx -\tilde{x}/|G|, \tag{4.11}$$

which holds throughout $\tilde{x}_- < \tilde{x} < \tilde{x}_+$ except near the end point $\tilde{x}_-$ where $\tilde{S}$ must have a zero, and near the end point $\tilde{x}_+ (= (8A_c/|G|)^{\frac{1}{2}} \ll 1)$ where a double zero is required. Since the maximum value of $\tilde{S}$ is unity it follows from (4.11) that $\tilde{x}_- = -|G|$, and from (4.7) that the solution in this neighbourhood is

$$\tilde{S} \approx \tanh\left(\frac{|G|^2}{8}(\tilde{x}+|G|)\right) \tag{4.12}$$

in the region with $0 < |G|^2(\tilde{x}+|G|) = O(1)$. In the neighbourhood of $\tilde{x}_+$ it is necessary to solve the full scaled equation (4.9) with $\tilde{R}(1) = 0$ and $A = A_c$; this solution has $\tilde{R}(\tilde{X}) \approx -\tilde{X}$ as $\tilde{X} \to -\infty$ and achieves a match with (4.11). The corresponding solution for $\tilde{u}$ is, from (4.4) with $D_+ = D_- = -\frac{1}{8}\tilde{x}_+^2$, the value of D_- being quoted in anticipation of the results of §4.3, since $\tilde{x}_+ \ll 1$,

$$\tilde{u} = 0, \quad \text{for } \tilde{x} < -|G|, \quad \tilde{x} > 0, \tag{4.13}$$

i.e. outside the collapsed eddy, and

$$\tilde{u} = -\tfrac{1}{2}\tilde{x}(1-\operatorname{sgn} G), \tag{4.14}$$

when $-|G| < \tilde{x} < 0$. The corrections in the neighbourhoods of the leading and trailing edges follow from those for $\tilde{S}$. Thus for large positive G (negative vorticity) the fluid velocity is essentially zero inside the eddy except for a forward pulse near the sharp leading edge; this forward pulse arises because, from (4.6), it follows that $\tilde{u}(\tilde{x}_-) = -\frac{1}{2}\tilde{x}_-$. The discontinuity to the right is accommodated by the solution (4.12) for $\tilde{S}$ with $\tilde{u} = -\frac{1}{2}(\tilde{x}+G\tilde{S})$, and that to the left by the adjustment region analysis of §4.3. When G is large and negative the velocity is again in the positive direction and decreases linearly from the large value $-\tilde{x}_-$ near the sharp leading edge to zero at the cusped trailing edge. There is again a double discontinuity at the leading edge; $\tilde{u}$ is reduced from $-\tilde{x}_-$ to $-\frac{1}{2}\tilde{x}_-$ in the region $0 < |G|^2(\tilde{x}+|G|) = O(1)$ where (4.12) holds, and then from $-\frac{1}{2}\tilde{x}_-$ to zero (in this large $|G|$ situation) in the adjustment region of §4.3.

The solution we have described in this section is illustrated in figure 1, where we have anticipated the results of §§4.2, 4.3 that $\tilde{u}$ is continuous at the cusped downstream end, and has a determinate discontinuity at the sharp upstream end, with the result that D_+, D_- in (4.4) are both equal to $-\frac{1}{8}\tilde{x}_+^2$. In figure 1(*a*) we take $G = 0$ and the maximum value of $\tilde{S}$ to be unity which determines $\tilde{x}_+$ as $6^{\frac{1}{3}}$. For figure 1(*b*) we have chosen $A = 0.3$ for a numerical integration of (4.9) which, upon scaling so that again the maximum value of $\tilde{S}$ is unity, gives $\tilde{x}_+ \approx 1.76$ and $|G| \approx 0.78$. The velocity profile $\tilde{u}$ is plotted for G both positive and negative and we see that A (and $|G|$) is sufficiently large for $\tilde{u}$ to depart from its zero-vorticity limit value of $-\frac{1}{2}\tilde{x}$ for $\tilde{x}_- < \tilde{x} < \tilde{x}_+$; the value of $\tilde{x}_-$ is -3.66. In figure 1(*c*), $A = 0.59$, which is very near to its critical maximum A_c, and we see that the characteristics of the limit solution described above are being attained although the value of $|G|$ is only 3.59 as yet. The values of $\tilde{x}_+, \tilde{x}_-$ are approximately 1.15 and -4.07 respectively, also some way from their limiting values of zero and $-|G|$.

At the cusped downstream end we obtain no further information from the adjustment region but must examine the solution to confirm that the smoothing is possible. At the sharp upstream end the solution in the adjustment region determines the discontinuity in $\tilde{u}$ and the constant D_- in (4.4*b*).

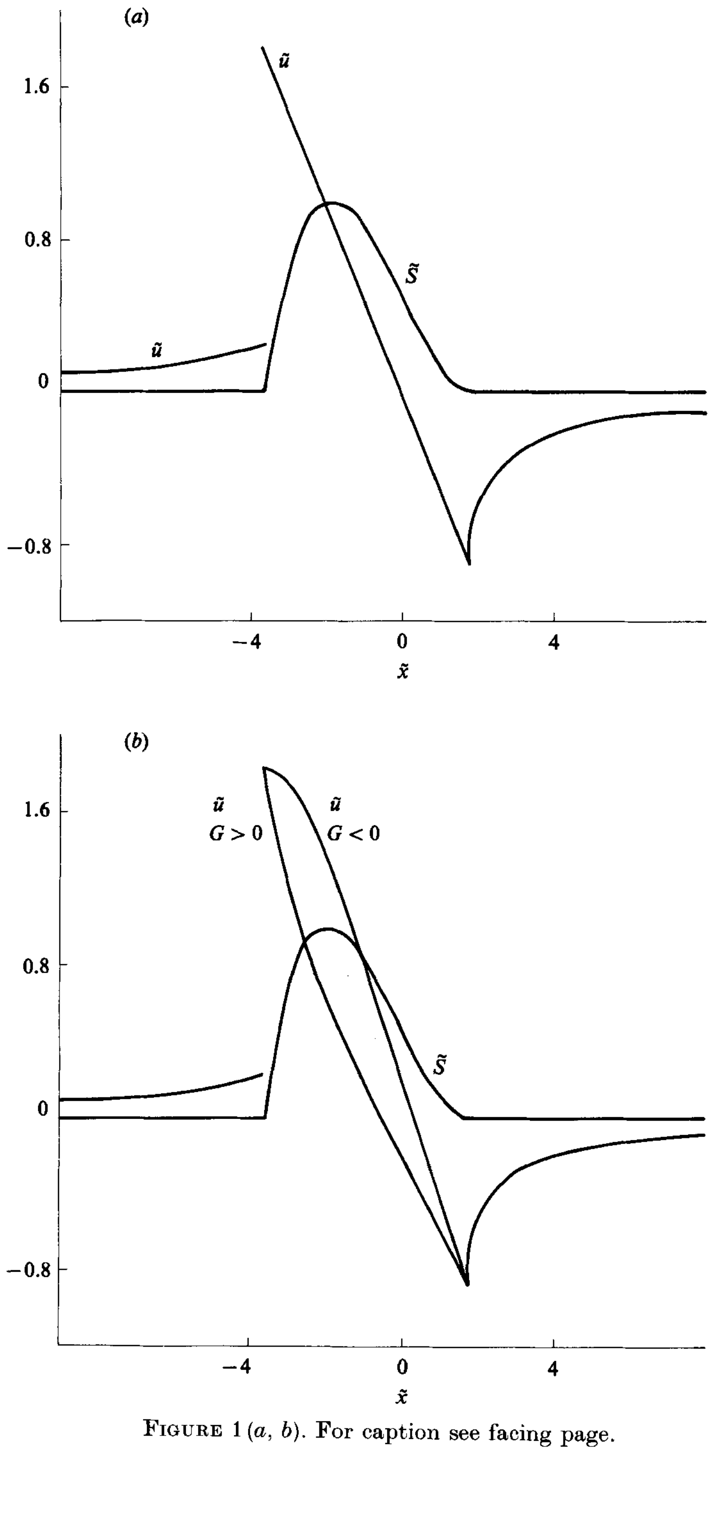

FIGURE 1 (a, b). For caption see facing page.

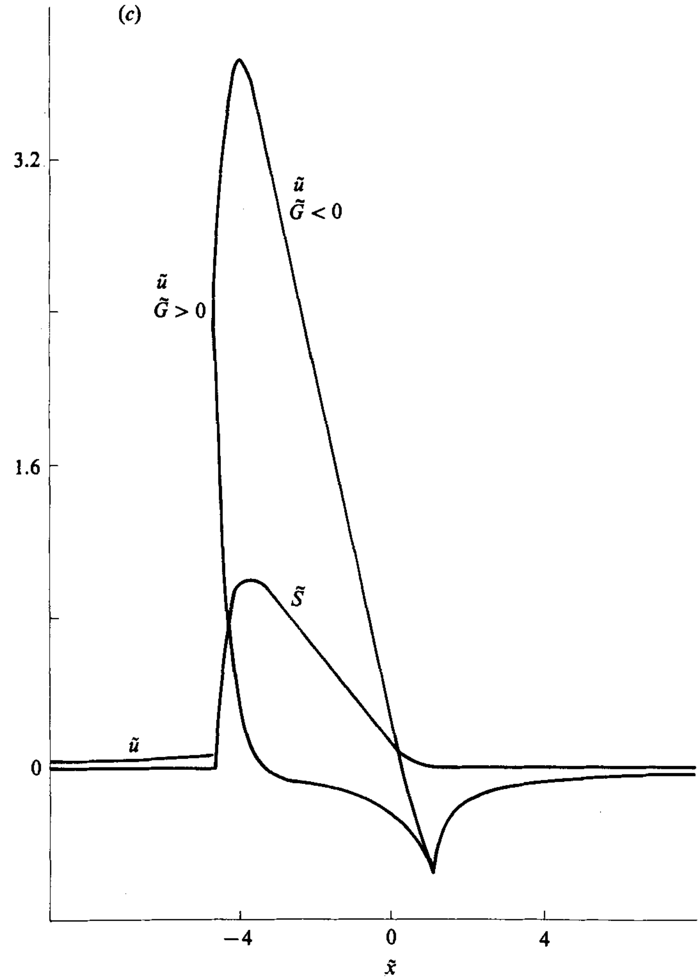

FIGURE 1. (*a*) Supersonic eddy with $n = \frac{1}{2}$ and $G = 0$: values of $\tilde{S}, \tilde{u}$. (*b*) As (*a*) but with $|G| = 0.78$. (*c*) As (*a*) but with $|G| = 3.59$.

4.2. *The downstream adjustment region at $\tilde{x} = \tilde{x}_+$: the cusped end*

To discuss the adjustment region we must return to the full unsteady equations (2.4), (2.5) together with (2.2). We examine the neighbourhood of $\tilde{x} = \tilde{x}_+$ by setting

$$x - x_s = (t_s - t)^{\frac{1}{2}}\tilde{x}_+ + \delta\xi \quad \text{with } \delta \ll (t_s - t)^{\frac{1}{2}}, \tag{4.15}$$

and

$$S \approx \delta^2 (t_s - t)^{-\frac{3}{2}}\bar{S}(\xi), \quad u \approx -\tfrac{1}{2}\tilde{x}_+(t_s - t)^{-\frac{1}{2}} + \delta^{\frac{1}{2}}(t_s - t)^{-\frac{3}{4}}\bar{u}(\xi), \tag{4.16}$$

in (2.4), (2.5), (2.2), the powers of $(t_s - t)$ following from the transformation (3.1). It is expected that the precise dependence of δ upon t is determined by the initial conditions. The double zero of $\tilde{S}$ at $\tilde{x}_+$ leads to the factor δ^2 in S, and the term $O(\delta^{\frac{1}{2}})$ in u arises because $\tilde{u}$ in (4.4*a*) has a square-root singularity at $\tilde{x}_+$ when D_+ is chosen so that $\tilde{u}$ is continuous at $\tilde{x}_+$, as is also implied in (4.16). The leading terms in (2.4), (2.5) are

$$-\tfrac{1}{4}\tilde{x}_+ + \bar{u}\bar{u}' = -\bar{S}'', \quad (\bar{u}\bar{S})' = 0, \tag{4.17}$$

which integrate to give

$$\bar{u}\bar{S} = c_+, \quad \bar{S}' = d_+ + \tfrac{1}{4}\tilde{x}_+\xi - c_+^2/2\bar{S}^2, \tag{4.18}$$

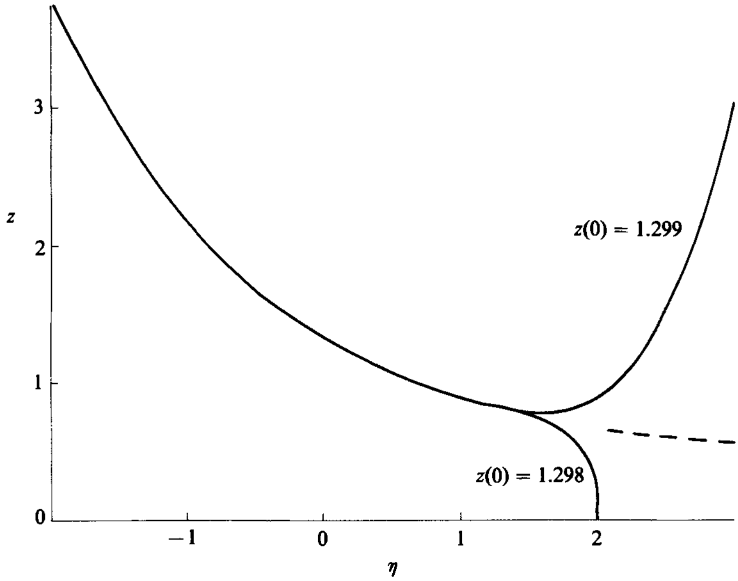

FIGURE 2. Solutions of the supersonic adjustment equation (4.19).

where $c_+(>0), d_+$ are constant. The match with the outer solution (4.5) is achieved, regardless of the values of the constants c_+, d_+, provided that (4.18) has a solution in which $\bar{S} \approx \frac{1}{8}\tilde{x}_+\xi^2$ as $\xi \to -\infty$ and $\bar{S} \approx (2c_+^2/\tilde{x}_+\xi)^{-\frac{1}{2}}$ as $\xi \to +\infty$ with $\bar{S} > 0$. This is equivalent to requiring a solution of

$$z' = \eta - z^{-2} \quad \text{with } z \approx \eta^{-\frac{1}{2}}, \tfrac{1}{2}\eta^2 \quad \text{as } \eta \to \pm\infty, \tag{4.19}$$

in which z remains positive over $(-\infty, \infty)$. Differentiation of (4.19) and examination of the solutions of the resulting equation in the (z', z)-plane indicates that the required solution exists and is unique. It has, in addition, been obtained by a Runge–Kutta integration and we deduce that $z(0) \approx 1.30$. Figure 2 illustrates the solutions with $z(0) = 1.298$ and $z(0) = 1.299$ which, respectively, tend to zero at a finite value of η, and increase like η^2 for $\eta \gg 1$. The required solution lies between these and is indicated in figure 2 by the dashed curve calculated from the first two terms of its asymptotic expansion for $\eta \gg 1$.

4.3. *The upstream adjustment region at $\tilde{x} = \tilde{x}_-$: the sharp end*

This adjustment region determines the discontinuity in $\tilde{u}$ at the sharp leading edge of the collapsed eddy, i.e. it determines the constant D_- in (4.4*b*). We return again to the unsteady equations (2.4), (2.5) and write

$$x - x_s = -2\tilde{x}_+(t_s - t)^{\frac{1}{2}} + \Delta\xi \quad \text{with } \Delta \ll (t_s - t)^{\frac{1}{2}}, \tag{4.20}$$

and

$$S = \Delta(t_s - t)^{-1}\bar{S}(\xi), \quad u = (t_s - t)^{-\frac{1}{2}}\bar{u}(\xi), \tag{4.21}$$

analogously to (4.15), (4.16). This time the leading-order terms are

$$-\tilde{x}_+\bar{S}' + (\bar{u}\bar{S})' = 0, \quad -\tilde{x}_+\bar{u}' + \bar{u}\bar{u}' = -\bar{S}'', \tag{4.22}$$

so that

$$(-\tilde{x}_+ + \bar{u})\bar{S} = -c_-, \quad -\tilde{x}^+\bar{u} + \tfrac{1}{2}\bar{u}^2 = -\bar{S}' + d_-, \tag{4.23a, b}$$

where c_-, d_- are constants. Elimination of $\bar{u}$ gives

$$\bar{S}' = (d_- + \tfrac{1}{4}\tilde{x}_+^2) - c_-^2/2\bar{S}^2. \tag{4.24}$$

Now as $\xi \to \infty$ this must lead to a match with (4.5), i.e. $\bar{S} \approx (d_- + \frac{1}{4}\tilde{x}_+^2)\,\xi$ where $d_- + \frac{1}{4}\tilde{x}_+^2 = \frac{9}{24}\tilde{x}_+^2$, and as $\xi \to -\infty$ where $\bar{S} = o(\xi)$ (in fact where, from (4.24),

$$\bar{S} \to [c_-^2/2(d_- + \tfrac{1}{4}\tilde{x}_+^2)]^{\frac{1}{2}})$$

we must have a match between $\bar{u}$ in (4.23*a*) and $\tilde{u}$ in (4.4*b*) as $\tilde{x} \to -2\tilde{x}_+$. This finally gives $D_- = -\frac{1}{8}x_+^2$, a result which implies that the decrease in $\tilde{u}$ as $\tilde{x}$ goes from $-2\tilde{x}_+ + 0$ to $-2\tilde{x}_+ - 0$ is $(2\lambda)^{\frac{1}{2}}$ where $\lambda = \tilde{S}'(-2\tilde{x}_+ + 0)$, i.e. the slope of the leading edge of the eddy. We shall find that the same relationship holds in a similar situation at other values of n.

The solution to equation (4.24) is

$$\frac{\xi - \xi_0}{\beta} = \frac{\bar{S}}{\alpha} + \tfrac{1}{2}\ln\left(\frac{\bar{S} - \alpha}{\bar{S} + \alpha}\right), \tag{4.25}$$

where $$\alpha = [c_-^2/2(d_- + \tfrac{1}{4}\tilde{x}_+^2)]^{\frac{1}{2}}, \quad \beta = \alpha(d_- + \tfrac{1}{4}\tilde{x}_+^2)^{-1}, \tag{4.26}$$

and ξ_0 is an arbitrary constant, which satisfies all the requirements.

When the vorticity constant G is non-zero the adjustment regions at the ends of the eddy are exactly the same, because the term GSS_x in (2.7) is unimportant at leading order. Essentially the difference is that (4.1) does not hold when G is non-zero, and we cannot take a fundamental solution, say with $\tilde{x}_+ = 1.0$ or $\max \tilde{S} = 1.0$, and derive all others from it using (4.1) for arbitrary $A > 0$. If G is non-zero the solution obtained from such a fundamental one would be a solution, but not for the same value of G, and as noted above there would be solutions for $\tilde{x}_+^2\,|G| < 4.8$ only.

4.4. *The value* $n = \frac{2}{3}$

The other limit, i.e. $n = \frac{2}{3}$, of the range of n under consideration, is such that the eddy boundary $\tilde{S}$ may tend to a non-zero limit either upstream or downstream, or both. In view of the transformation (4.1) we may restrict ourselves to considering those solutions with $\tilde{S}(-\infty) = 1.0$ and $\tilde{S}(\infty) < 1.0$ or conversely. In this situation a linearization about the solution $\tilde{S} \equiv 1.0, \tilde{u} \equiv 0$ is possible and if we write

$$\tilde{S} = 1 + \epsilon \tilde{S}_1, \quad \tilde{u} = \epsilon \tilde{u}_1 \tag{4.27}$$

for arbitrary small constant amplitude ϵ then (3.3), (3.5) become, on neglecting the nonlinear terms,

$$\tfrac{2}{3}\tilde{x}\tilde{S}_1' + \tilde{u}_1' = 0, \quad \tfrac{1}{3}\tilde{u}_1 + \tfrac{2}{3}\tilde{x}\tilde{u}_1' + \tilde{S}_1'' = 0, \tag{4.28}$$

from which we find that the solution for $\tilde{u}_1$ is

$$\tilde{u}_1 = U(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{4}{27}\tilde{x}^3) \tag{4.29}$$

where U is the confluent hypergeometric function in the usual notation. The analytic continuation to negative $\tilde{x}$ must be made using formulae (13.1.9,10,27) of Abramowitz & Stegun (1965) with $\arg \tilde{x}^3 = 3\pi$. For $\tilde{x} \gg 1, \tilde{u} = O(\tilde{x}^{-\frac{1}{2}})$ as required, and for $\tilde{x} \ll -1, \tilde{u} = O\,(\exp \frac{4}{27}\tilde{x}^3)$ and decays exponentially, although in general the nonlinear solutions of (3.3), (3.5) will have $\tilde{u} = O(|\tilde{x}|^{-\frac{1}{2}})$ as $\tilde{x} \to -\infty$ also.

The solution (4.29) is useful for checking the numerical scheme that is required for the nonlinear system by verifying the accuracy of solutions at small but non-zero values of ϵ. For example, if $\tilde{u}_1(0) = \tilde{\alpha}$, then integrals of the confluent hypergeometric functions may be used to give

$$\tilde{S}_1'(0) = 2^{-\frac{1}{3}}\tilde{\alpha}\Gamma(\tfrac{5}{6})\,\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{1}{6})\,\Gamma(\tfrac{2}{3}), \tag{4.30}$$

and $$\tilde{S}_1(\infty) = 2^{-\frac{4}{3}}\pi^{-\frac{1}{2}}\tilde{\alpha}\Gamma(\tfrac{5}{6})\,\Gamma(\tfrac{1}{3}), \quad \tilde{S}_1(-\infty) = -2\tilde{S}_1(\infty), \tag{4.31}$$

for the case with $\tilde{S}_1(0 = 0$. A constant may be added to $\tilde{S}_1$ to make $\tilde{S}_1$ vanish at $\tilde{x}$ equal to plus or minus infinity if required. The different behaviour of $\tilde{u}$ at $\tilde{x} = \pm\infty$ and the differing values of $|\tilde{S}_1(\pm\infty)|$ are further illustrations of the non-reversibility of this supersonic flow.

The other solutions of (3.3), (3.5) with $n = \frac{2}{3}$ must be found numerically and this was achieved by first setting $\tilde{S}(-\infty) = 1.0$ and gradually reducing $\tilde{S}(\infty)$ from 1.0 (i.e. gradually departing from the linearized solution). Subsequently we solved the complementary problem with $\tilde{S}(\infty) = 1.0$ and $\tilde{S}(-\infty) < 1.0$. In the first of these investigations we found that it is possible to reduce $\tilde{S}(\infty)$ to 0^+ and that the limiting solution is very similar to the cusped downstream collapse with $n = \frac{1}{2}$ described earlier. For $\tilde{x} > \tilde{x}_+ > 0$, $\tilde{S}$ is identically zero; at $\tilde{x} = \tilde{x}_+$ the value of $\tilde{u}$ is $-\frac{2}{3}\tilde{x}_+, \tilde{u}$ is continuous through $\tilde{x}_+$ although its derivative is discontinuous, and $\tilde{S}(\tilde{x}_+ - 0) = \tilde{S}'(\tilde{x}_+ - 0) = 0$ so that both $\tilde{S}$ and $\tilde{S}'$ are continuous. In the second investigation with $\tilde{S}(\infty) = 1.0$ it is possible to reduce $\tilde{S}(-\infty)$ to 0^+, and the limiting solution resembles the collapse at the sharp upstream end of the eddy with $n = \frac{1}{2}$. The point of collapse is $\tilde{x} = \tilde{x}_- < 0$ where $\tilde{S}$ is continuous, although $\tilde{S}'$ and $\tilde{u}$ are discontinuous with $\tilde{u}(\tilde{x}_- + 0) = -\frac{2}{3}\tilde{x}_-$. Figures 3 and 4 illustrate the phenomena for $\tilde{S}(-\infty) = 1.0$ with $\tilde{S}(\infty) = 0.5, 0.1, 0.0$, and for $\tilde{S}(\infty) = 1.0$ with $\tilde{S}(-\infty) = 0.5, 0.1, 0.0$ respectively.

The analysis of the adjustment regions may be carried through exactly as for $n = \frac{1}{2}$. For $\tilde{x} > \tilde{x}_+$ in the first investigation, and for $\tilde{x} < \tilde{x}_-$ in the second, we have, from (3.3*b*) with $\tilde{S} \equiv 0$,

$$\tilde{u}^2(\tilde{u} + \tilde{x}) = C_\pm, \tag{4.32}$$

where the constant $C_+ = \frac{4}{27}\tilde{x}_+^3$, since $\tilde{u}$ is continuous at $\tilde{x}_+$. The root for $\tilde{u}$ must be chosen so that $\tilde{u} \to 0$ as $\tilde{x} \to \pm\infty$ respectively in the two problems; thus $C_+ > 0$, $C_- < 0$. To examine the adjustment region of the cusped collapse in the first problem we make the transformation (4.15), (4.16), although with the appropriate powers of $t_s - t$ that follow from (3.1) with $n = \frac{2}{3}$, and obtain equations equivalent to (4.18); thus the adjustment region is once again passive and the existence of the solution serves to confirm the structure. However for the upstream collapse in the second problem the discontinuity in $\tilde{u}$ is determined by the adjustment region, the equation for which is analogous to (4.24), and the same result holds as was noted there – there is a decrease in $\tilde{u}$ as $\tilde{x}$ goes from $\tilde{x}_- + 0$ to $\tilde{x}_- - 0$ of magnitude $(2\lambda)^{\frac{1}{2}}$ where, as there, $\lambda = \tilde{S}'(\tilde{x}_- + 0)$. This implies that C_- in (4.32) is given by

$$C_- = (\tfrac{2}{3}\tilde{x}_- + (2\lambda)^{\frac{1}{2}})^2 (\tfrac{1}{3}\tilde{x}_- - (2\lambda)^{\frac{1}{2}}). \tag{4.33}$$

The limiting solutions shown in figures 3, 4 were plotted using numerical solutions of (3.3), (3.5) for the regions where $\tilde{S}$ was non-zero, and (4.32) for $\tilde{u}$ for $\tilde{x} > \tilde{x}_+$ and $\tilde{x} < \tilde{x}_-$ respectively. For the cusped collapse the finite-difference scheme reproduced the square-root singularity in $\tilde{u}$ at $\tilde{x} = \tilde{x}_+ + 0$ very well (no doubt because $\tilde{S}, \tilde{S}', \tilde{u}$ are continuous there) but could not predict well the jump discontinuity in $\tilde{u}$ at $\tilde{x}_-$. We estimate that $\tilde{x}_+ \approx 1.89$, $\tilde{x}_- \approx -2.00$, and $\lambda \approx 0.68$.

Another point of interest is the small overshoot of $\tilde{S}$ over its value of unity at $\tilde{x} = -\infty$. This is of about 1% and is invisible in figure 3(*a*), but its presence can be deduced by the maximum of $\tilde{u}$ in figure 3(*b*), since $\tilde{S}$ and $\tilde{u}$ are stationary together.

Thus when $n = \frac{2}{3}$ there are many more solutions than there are when $n = \frac{1}{2}$, in which case there is only the one solution we described in §4.1 together with those to be obtained from it by use of the transformation (4.1). We shall, by taking $n = 0.6$, support the conjecture that as n decreases the solutions get sparser until finally the doubly-collapsed situation with $n = \frac{1}{2}$ is attained.

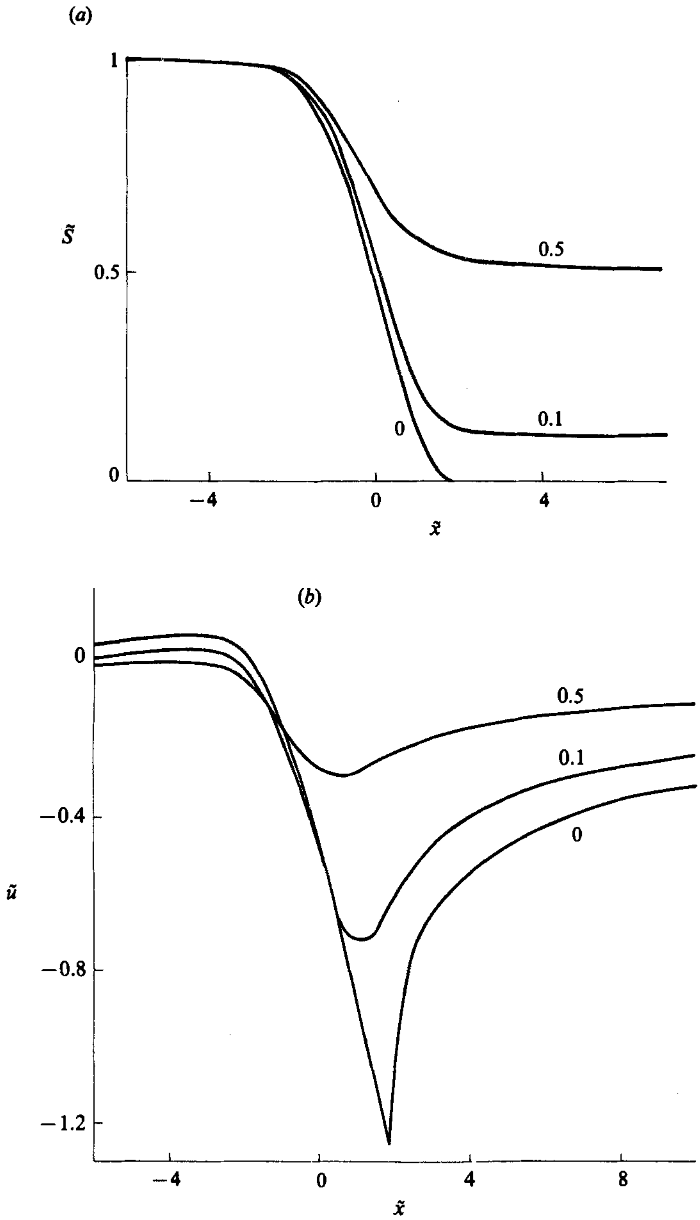

FIGURE 3. (*a*) Supersonic eddy with $n = \frac{2}{3}$: values of $\tilde{S}$ for $\tilde{S}(-\infty) = 1.0$ with $\tilde{S}(\infty) = 0.5, 0.1, 0$. (*b*) Values of $\tilde{u}$ corresponding to $\tilde{S}$ in (*a*).

4.5. *The value* $n = 0.6$

When $n = 0.6$ the solutions for $\tilde{S}$ in (3.3), (3.5) must be such that $\tilde{S}$ decays algebraically as $|\tilde{x}| \to \infty$. The decay is $O(|\tilde{x}|^{-\frac{1}{3}})$ with $\tilde{u} = O(|\tilde{x}|^{-\frac{2}{3}})$. The algebraic decay was accommodated in the numerical integration by taking $\tan^{-1}\tilde{x}$ over $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ as the independent variable; this was also done for $n = \frac{2}{3}$. Now $\tilde{S}$ has a maximum at a finite value of $\tilde{x}$ and we took this to be unity. It was found that if $\tilde{S}(0) = 1.0$ then $\tilde{S}$ is non-zero for all finite $\tilde{x}$, but if this maximum is moved sufficiently far to the left a cusped downstream collapse occurs with $\tilde{S} \equiv 0$ for $\tilde{x} > \tilde{x}_+$ for some $\tilde{x}_+ > 0$.

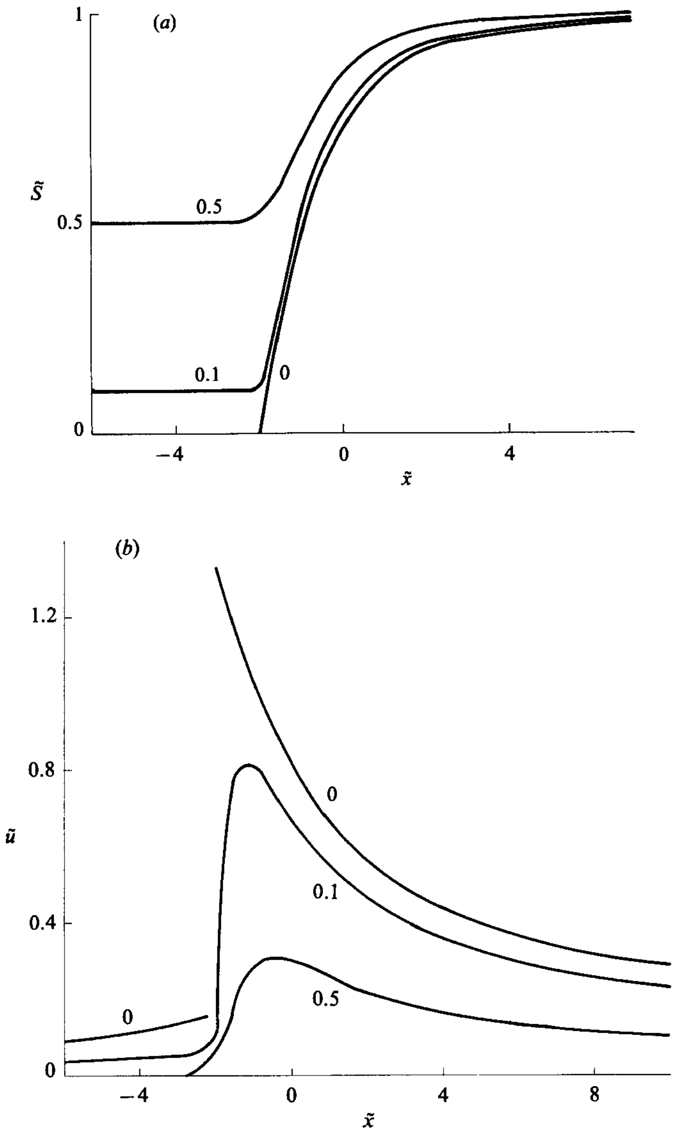

FIGURE 4. (a) Supersonic eddy with $n = \frac{2}{3}$: values of $\tilde{S}$ for $\tilde{S}(\infty) = 1.0$ with $\tilde{S}(-\infty) = 0.5, 0.1, 0$. (b) Values of $\tilde{u}$ corresponding to $\tilde{S}$ in (a).

Similarly, if the maximum is moved sufficiently far to the right a sharp upstream collapse is evident with $\tilde{S} \equiv 0$ for $\tilde{x} < \tilde{x}_- < 0$. The phenomenon is illustrated in figure 5 (a, b) where we have drawn $\tilde{S}, \tilde{u}$ for the case when $\tilde{S}$ has unit maximum at the origin, and for the two limiting situations. Equation (4.32) is, for general n, to be replaced by

$$|\tilde{u}|^n \, |\tilde{u}+\tilde{x}|^{1-n} = C_{\pm}, \tag{4.34}$$

and the adjustment regions may be analysed exactly as before. Again the downstream adjustment region is passive with $\tilde{S}, \tilde{S}'$ and $\tilde{u}$ continuous through $\tilde{x} = \tilde{x}_+$; the value of $\tilde{u}(\tilde{x}_+)$ is $-n\tilde{x}_+$. The upstream adjustment region determines the jump in $\tilde{u}$ from $-n\tilde{x}_-$ at $\tilde{x} = \tilde{x}_- + 0$ to $-n\tilde{x}_- - (2\lambda)^{\frac{1}{2}}$ at $\tilde{x} = \tilde{x}_- - 0$, where again λ is the slope of the

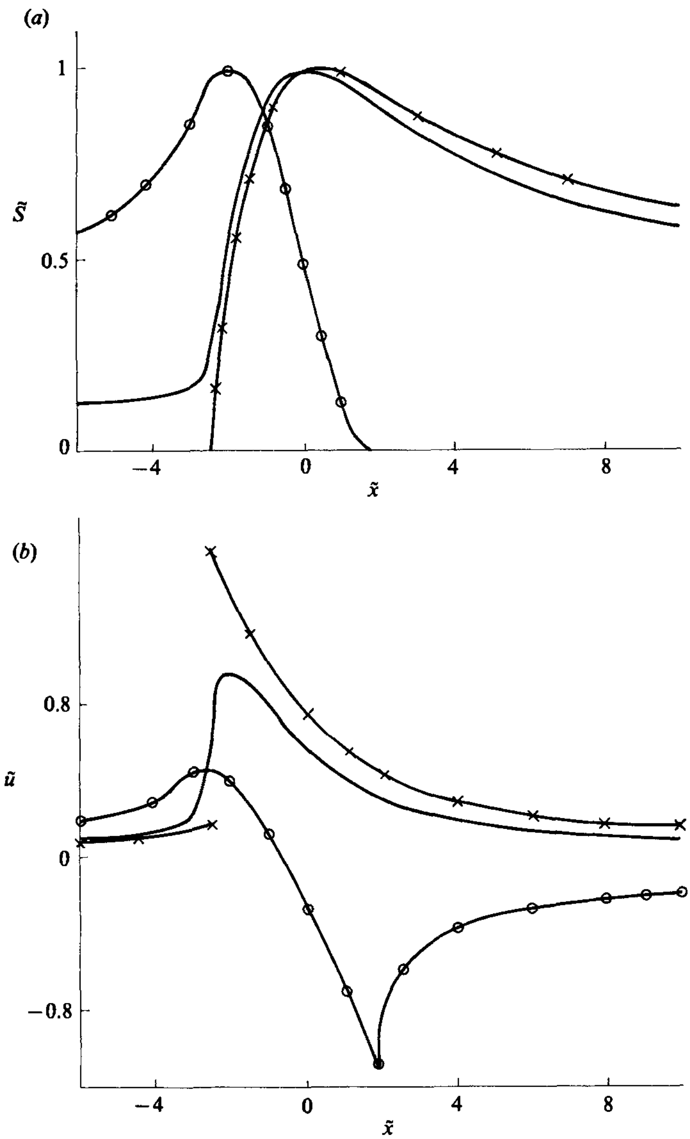

FIGURE 5. (a) Supersonic eddy with $n = 0.6$: values of $\tilde{S}$ for ——, eddy with unit maximum at the origin; ○, eddy with cusped trailing edge; and ×, eddy with sharp leading edge. (b) Values of $\tilde{u}$ corresponding to $\tilde{S}$ in (a).

eddy boundary at $\tilde{x}_- + 0$. The eddy with the cusped downstream collapse has its maximum height at $\tilde{x} \approx -1.98$, and $\tilde{x}_+ \approx 1.8$. The eddy with the sharp upstream collapse has its maximum at $\tilde{x} \approx 0.35$ with $\tilde{x}_- \approx -2.5$ and $\lambda \approx 0.9$.

5. Subsonic eddy-breakdown solutions

When the free stream is subsonic we must solve (3.3) together with the Cauchy integral relation (3.4) rather than (3.5). We consider the same values of n, i.e. $\frac{1}{2} \leqslant n \leqslant \frac{2}{3}$, so that $\tilde{S}, \tilde{u}$ must tend to zero as $|\tilde{x}| \to \infty$ (except for $n = \frac{2}{3}$) with

$$\tilde{S} = O(|\tilde{x}|^{(3n-2)/n})$$

and $\tilde{u} = O(|\tilde{x}|^{(n-1)/n})$ as before. The transformation under which the system is invariant is now, to replace (4.1),

$$(\tilde{S}, \tilde{u}, \tilde{p}, \tilde{x}) \to (|A|^3 \tilde{S}, A\tilde{u}, A^2\tilde{p}, A\tilde{x}), \tag{5.1}$$

for any constant A, so that this subsonic flow is reversible as might be expected. This means that we shall not have to examine as many solutions as we did in §4. The relationship (4.2) is still valid and so restriction to values of n greater than or equal to $\frac{1}{2}$ is justified.

As in the supersonic situation an analytic solution is available if $n = \frac{1}{2}$, a linearized solution if $n = \frac{2}{3}$, and otherwise solutions must be obtained numerically. Collapsed eddies are again a feature of the model, and to illustrate these we consider first $n = \frac{1}{2}$ in which, as before, a doubly-collapsed eddy appears to be the only possibility.

5.1. *The value* $n = \frac{1}{2}$

When $n = \frac{1}{2}$ equations (3.3) may be integrated to give

$$(\tfrac{1}{2}\tilde{x} + \tilde{u})\tilde{S} = C, \quad \tfrac{1}{2}(\tilde{x} + \tilde{u})\tilde{u} + \tilde{p} = D, \tag{5.2a, b}$$

which are the same as (4.3) except that $\tilde{p}$ is related to $\tilde{S}$ by (3.4), and has not been replaced in (5.2*b*). For the same reasons as for the supersonic situation we take $C = 0$, and satisfy (5.2*a*) by assuming that $\tilde{S} = 0$ for $\tilde{x} < \tilde{x}_-$ and for $\tilde{x} > \tilde{x}_+$, whereas $\tilde{u} = -\frac{1}{2}\tilde{x}$ for $\tilde{x}_- < \tilde{x} < \tilde{x}_+$. It then follows from (5.2*b*) that $\tilde{p} = \frac{1}{8}\tilde{x}^2 + D$ when $\tilde{x}_- < \tilde{x} < \tilde{x}_+$. Since an interpretation of the relationship (3.4) is that $\tilde{p}$ and $\tilde{S}'$ are the values at $\tilde{y} = 0$ of the real and imaginary parts of a function $Q(\tilde{z})$ of the complex variable $\tilde{z} = \tilde{x} + i\tilde{y}$, and either one or other is known over the whole real axis, the solution is now available. It is found that if $\tilde{S}$ is to be continuous at $\tilde{x}_-, \tilde{x}_+$ then it is necessary that $\tilde{x}_- = -\tilde{x}_+$ and

$$Q(\tilde{z}) = \tfrac{1}{8}[\tilde{z}^2 - \tfrac{1}{2}\tilde{x}_+^2 - \tilde{z}(\tilde{z}^2 - \tilde{x}_+^2)^{\frac{1}{2}}]. \tag{5.3}$$

Thus for $|\tilde{x}| < \tilde{x}_+$ we have

$$\tilde{S} = \tfrac{1}{24}(\tilde{x}_+^2 - \tilde{x}^2)^{\frac{3}{2}}, \quad \tilde{u} = -\tfrac{1}{2}\tilde{x}, \quad \tilde{p} = \tfrac{1}{8}(\tilde{x}^2 - \tfrac{1}{2}\tilde{x}_+^2), \tag{5.4}$$

and for $|\tilde{x}| > \tilde{x}_+$

$$\left.\begin{aligned} \tilde{S} = 0, \quad \tilde{u} &= -\tfrac{1}{2}\tilde{x} + \tfrac{1}{2}(\operatorname{sgn}\tilde{x})\,|\tilde{x}|^{\frac{1}{2}}(\tilde{x}^2 - \tilde{x}_+^2)^{\frac{1}{4}}, \\ \tilde{p} &= \tfrac{1}{8}[\tilde{x}^2 - \tfrac{1}{2}\tilde{x}_+^2 - |\tilde{x}|(\tilde{x}^2 - \tilde{x}_+^2)^{\frac{1}{2}}]. \end{aligned}\right\} \tag{5.5}$$

We have, therefore, a symmetric cusped collapsed eddy with infinite curvature at the end points. The velocity $\tilde{u}$ and pressure $\tilde{p}$ are continuous there but have discontinuous derivatives. As in the supersonic situation the adjustment region in the neighbourhood of the singularity is analysed by returning to the full unsteady equations (2.4), (2.5) together with (2.1).

For the adjustment region in the neighbourhood of $\tilde{x} = \tilde{x}_+$ we make the transformation (4.15) and write, analogously to (4.16),

$$\begin{aligned} S \approx \delta^{\frac{3}{2}}(t_s - t)^{-\frac{5}{4}}\bar{S}(\xi), \quad u &= -\tfrac{1}{2}\tilde{x}_+(t_s - t)^{-\frac{1}{2}} + \delta^{\frac{1}{4}}(t_s - t)^{-\frac{5}{8}}\bar{u}(\xi), \\ p &= \tfrac{1}{16}\tilde{x}_+^2(t_s - t)^{-1} + \delta^{\frac{1}{2}}(t_s - t)^{-\frac{5}{4}}\bar{p}(\xi), \end{aligned} \tag{5.6}$$

with the result that, to leading order,

$$(\bar{u}\bar{S})' = 0, \quad \bar{u}\bar{u}' = -\bar{p}', \quad \bar{p}' = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{S}''(\eta)}{\xi - \eta}\,d\eta. \tag{5.7a–c}$$

In (5.7) the Cauchy integral relationship between $\bar{p}$ and $\bar{S}$ is written in differentiated form to ensure convergence of the integral. From (5.7*a, b*) we obtain, on integration,

$$\bar{u}\bar{S} = C_+, \quad \bar{p} = D_+ - C_+^2/2\bar{S}^2, \tag{5.8}$$

where C_+, D_+ are constants. As $\xi \to \pm\infty$ the solutions of (5.7), (5.8) must match with the outer solution (5.4), (5.5) as $\tilde{x} \to \tilde{x}_+$. This implies that the following asymptotic forms must be attained as $|\xi| \to \infty$:

$$\bar{S} \approx \tfrac{1}{24}(2\tilde{x}_+)^{\frac{3}{2}}(-\xi)^{\frac{3}{2}} \quad \text{as } \xi \to -\infty, \tag{5.9}$$

and

$$\bar{u} \approx (\tfrac{1}{2}\tilde{x}_+)^{\frac{3}{4}}\xi^{\frac{1}{4}}, \quad \bar{p} \approx -\tfrac{1}{2}(\tfrac{1}{2}\tilde{x}_+)^{\frac{3}{2}}\xi^{\frac{1}{2}} \quad \text{as } \xi \to \infty. \tag{5.10}$$

Since $\bar{S}$ is non-negative it follows that the constant C_+ is positive.

Equation (5.7*c*) may be integrated in the form

$$\bar{p} + \tfrac{1}{2}(\tfrac{1}{2}\tilde{x}_+)^{\frac{3}{2}}\xi^{\frac{1}{2}}H(\xi) - D_+ = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{S}'(\eta) + \tfrac{1}{2}(\tfrac{1}{2}\tilde{x}_+)^{\frac{3}{2}}(-\eta)^{\frac{1}{2}}H(-\eta)}{\xi - \eta}\,\mathrm{d}\eta, \tag{5.11}$$

where $H(\xi)$ is the Heaviside step function, with the result that the integral equation controlling the adjustment region is obtained from (5.8), (5.11) as

$$\frac{1}{\bar{S}^2} - \xi^{\frac{1}{2}}H(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{S}'(\eta) + (-\eta)^{\frac{1}{2}}H(\eta)}{\xi - \eta}\,\mathrm{d}\eta, \tag{5.12}$$

from which the constants have been removed by a simple scaling.

The required solution of (5.12) has to satisfy the scaled form of (5.9), (5.10), i.e.

$$\begin{aligned} \bar{S} &= \tfrac{2}{3}(-\xi)^{\frac{3}{2}} + \Gamma(-\xi)^{\frac{1}{2}} + \surd 2(-\xi)^{-\frac{1}{4}} + o((-\xi)^{-\frac{1}{4}}) \quad \text{as } \xi \to -\infty, \\ \bar{S} &= \xi^{-\frac{1}{4}} + \tfrac{1}{4}\Gamma\xi^{-\frac{5}{4}} + o(\xi^{-\frac{5}{4}}) \quad \text{as } \xi \to \infty, \end{aligned} \tag{5.13}$$

where the arbitrary constant Γ reflects the fact that the equations (5.7) are unaffected by an origin shift. Direct solutions of (5.12), (5.13) were difficult to find, but the equivalent system

$$\left.\begin{aligned} \frac{1}{T^2} - \xi^{\frac{1}{2}} + \frac{\Gamma}{2(1+\xi^{\frac{1}{2}})} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{T'(\eta)}{\xi - \eta}\,\mathrm{d}\eta \quad (\xi > 0), \\ \frac{1}{(T+\bar{f})^2} + \frac{\Gamma}{2(1-\xi)} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{T'(\eta)}{\xi - \eta}\,\mathrm{d}\eta \quad (\xi < 0), \end{aligned}\right\} \tag{5.14}$$

where

$$T(\xi) = \bar{S}(\xi) - \bar{f}(\xi),$$

and $\bar{f}(\xi)$ is defined by

$$\bar{f}(\xi) = \{\tfrac{2}{3}(-\xi)^{\frac{3}{2}} + \Gamma[(-\xi)^{\frac{1}{2}} - \tan^{-1}(-\xi)^{\frac{1}{2}}]\}H(-\xi), \tag{5.15}$$

was solved successfully with the boundary condition $T(-\infty) = \tfrac{1}{2}\Gamma\pi$. To obtain solutions it proved desirable to solve for $1/T(\xi)$ and to take $\tan^{-1}\xi$ over $(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$ as the independent variable; the Cauchy integrals were dealt with by a technique suggested by Davis & Werle (1982). Solutions computed for different values of Γ were consistent in that they differed by an origin shift, and in figure 6 we plot the solution with $\Gamma = 0$ which has $\bar{S}(0) \approx 1.33$.

The solution at the leading edge of the collapsed eddy may be obtained from the above by changing the signs of $\bar{u}$ and ξ. For comparison with the corresponding

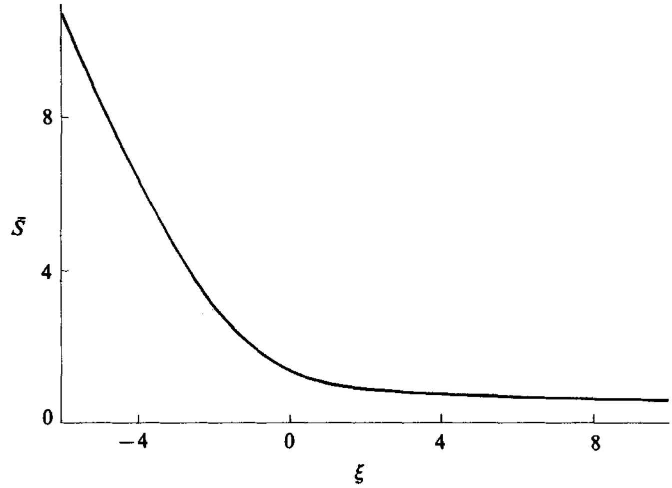

FIGURE 6. Solution of the subsonic adjustment equation (5.12).

quantities at the values $n = \frac{2}{3}, 0.6$ now to be considered we display, in figure 7, the values of $\tilde{S}, \tilde{u}$ given by (5.4), (5.5) with $\tilde{x}_+ = 24^{\frac{1}{3}}$, a choice made so that $\max \tilde{S}$ is unity. This may be compared also with the non-symmetric supersonic doubly-collapsed eddy in figure 1, of which the singularities at the sharp leading edge and at the cusped trailing edge are respectively more and less severe than at the cusped ends of this subsonic eddy.

Specific solutions for the subsonic eddy for non-zero vorticity with equation (3.3a) replaced by (3.6) have not been obtained but experience gained from the discussion of §4 for the corresponding supersonic flow suggests that the following description will apply. An integral of (3.6) gives, instead of (5.2a),

$$(\tfrac{1}{2}\tilde{x}+\tilde{u}+\tfrac{1}{2}G\tilde{S})\,\tilde{S} = 0, \tag{5.16a}$$

where as before the constant of integration is zero. The solution of (5.16a) coupled with (5.2b) is sought in split form, depending on whether $|\tilde{x}| \lessgtr \tilde{x}_+$ for some positive constant $\tilde{x}_+$. In the inner part where $-\tilde{x}_+ < \tilde{x} < \tilde{x}_+$

$$\tilde{u} = -\tfrac{1}{2}(\tilde{x}+G\tilde{S}), \tag{5.16b}$$

to satisfy (5.16a), and so (5.2b) yields

$$\tilde{p} = \tfrac{1}{8}(\tilde{x}^2-G^2\tilde{S}^2)+D, \tag{5.16c}$$

whereas $\tilde{S}(\tilde{x})$ remains unknown as yet. In the outer part, $|\tilde{x}| > \tilde{x}_+$, the solution of (5.16$a$), (5.2$b$) is

$$\tilde{S} \equiv 0, \quad \tilde{p} = -\tfrac{1}{2}(\tilde{x}+\tilde{u})\,\tilde{u}+D, \tag{5.16d}$$

but $\tilde{p}(\tilde{x})$ remains unknown. Then, equating the pressure in (5.16c) with that of the Cauchy–Hilbert integral in (3.4) we obtain the nonlinear integral equation

$$-\frac{1}{\pi}\int_{-\tilde{x}_+}^{\tilde{x}_+} \frac{\tilde{S}'(\eta)\,\mathrm{d}\eta}{\tilde{x}-\eta} = \tfrac{1}{8}(\tilde{x}^2-G^2\tilde{S}^2)+D, \tag{5.17a}$$

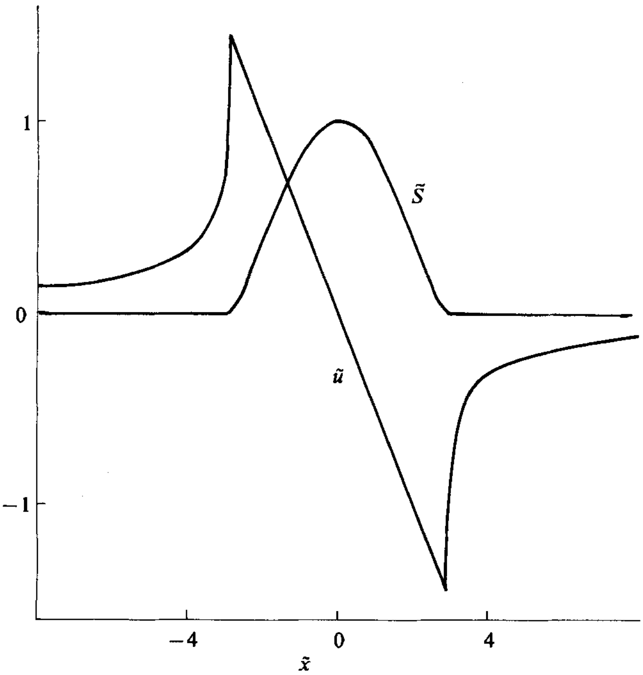

FIGURE 7. Subsonic eddy with $n = \frac{1}{2}$: values of $\tilde{S}, \tilde{u}$.

governing the function $\tilde{S}(\tilde{x})$ for $|\tilde{x}| < \tilde{x}_+$. Once $\tilde{S}$ is found from this equation the outer pressure and velocity distributions are given by (5.16*d*) with (3.4) and the inner ones follow from (5.16*b*, *c*). The constraints here are that $\tilde{p}, \tilde{u}$ are to be continuous at the junctions $\tilde{x} = \pm\tilde{x}_+$, and

$$\tilde{S} = \tilde{S}' = 0 \quad \text{at } \tilde{x} = \pm\tilde{x}_+. \tag{5.17b}$$

In contrast with the supersonic case of §4.1 for non-zero vorticity, symmetric solutions seem likely to exist for $\tilde{S}(\tilde{x})$ at all values of G. The limit $G \to 0$ reproduces the earlier solution (5.4), while for large G the $\tilde{x}^2$ term in the right-hand side of (5.17*a*) can be neglected, thus yielding an integrated form of the Benjamin–Ono equation: more specifically if we transform $(\tilde{x}, \tilde{S}) \to \tilde{x}_+(\tilde{x}, D\tilde{S})$ and set $D\tilde{x}_+^2 G^2 = 8\sigma^2$ then for large G, provided $\tilde{x}_+ \ll |G|^{-\frac{1}{2}}$, (5.17*a*) becomes the Benjamin–Ono equation solved by Smith (1985*b*), giving $\sigma = 1.368$ and a symmetric solution. The adjustment zones near $\tilde{x} = \pm\tilde{x}_+$ are affected little by the presence of the non-zero vorticity, we note, and in addition non-symmetric solutions more like the supersonic version may still be possible.

5.2. *The value* $n = \frac{2}{3}$

When $n = \frac{2}{3}$ solutions of (3.3), (3.4) may be obtained by setting $\tilde{S}(-\infty) = 1.0$ and $\tilde{S}(\infty) < 1.0$; other solutions then follow on use of the transformation (5.1). As in the supersonic regime a linearized solution about $\tilde{S} = 1, \tilde{u} = 0$ is available at this value of n. If, as in (4.27), we define $\epsilon\tilde{S}_1, \epsilon\tilde{u}_1, \epsilon\tilde{p}_1$ to be the perturbations, then

$$\tfrac{2}{3}\tilde{x}\tilde{S}_1' + \tilde{u}_1' = 0, \quad \tfrac{1}{3}\tilde{u}_1 + \tfrac{2}{3}\tilde{x}\tilde{u}_1' = -\tilde{p}_1', \tag{5.18}$$

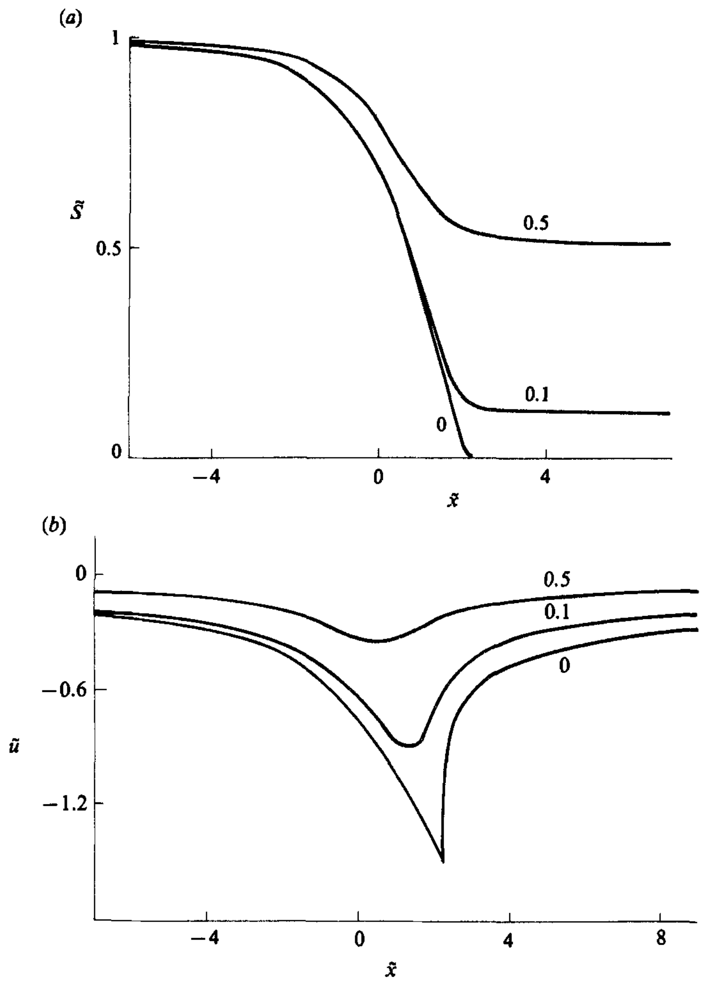

FIGURE 8(*a*). Subsonic eddy with $n = \frac{2}{3}$: values of $\tilde{S}$ for $\tilde{S}(-\infty) = 1.0$ with $\tilde{S}(\infty) = 0.5, 0.1, 0$. (*b*) Values of $\tilde{u}$ corresponding to $\tilde{S}$ in (*a*).

where $\tilde{p}_1$ is related to $\tilde{S}_1$ by the usual Cauchy integral. It may be shown, either by taking a Fourier transform or by direct substitution, that the solutions are proportional to

$$\left.\begin{aligned} \tilde{S}_1 &= \int_0^\infty e^{-\omega^{\frac{3}{2}}} \frac{\sin \omega\tilde{x}}{\omega}\, d\omega + \tfrac{1}{2}\pi, \\ \tilde{u}_1 &= \int_0^\infty e^{-\omega^{\frac{3}{2}}} \frac{\cos \omega\tilde{x}}{\omega^{\frac{1}{2}}}\, d\omega, \quad \tilde{p}_1 = -\int_0^\infty e^{-\omega^{\frac{3}{2}}} \sin \omega\tilde{x}\, d\omega, \end{aligned}\right\} \tag{5.19}$$

where the additive constant in $\tilde{S}_1$ has been chosen so that $\tilde{S}_1(-\infty = 0.0$. These solutions proved a useful check on the numerical procedure at small values of $1.0 - \tilde{S}(\infty)$; for example, in (5.19), $\tilde{u}_1(0) = 4\Gamma(\frac{4}{3}), \tilde{S}_1'(0) = \frac{4}{3}\Gamma(\frac{2}{3})$ and these could be verified immediately.

As in the discussion of §4.4 for supersonic breakdown with $n = \frac{2}{3}$, the value of $\tilde{S}(\infty)$ may be continuously decreased from unity and in figure 8(*a*, *b*) we present $\tilde{S}, \tilde{u}$ when $\tilde{S}(\infty) = 0.5, 0.1, 0$. The numerical techniques employed here and in §5.3 were similar

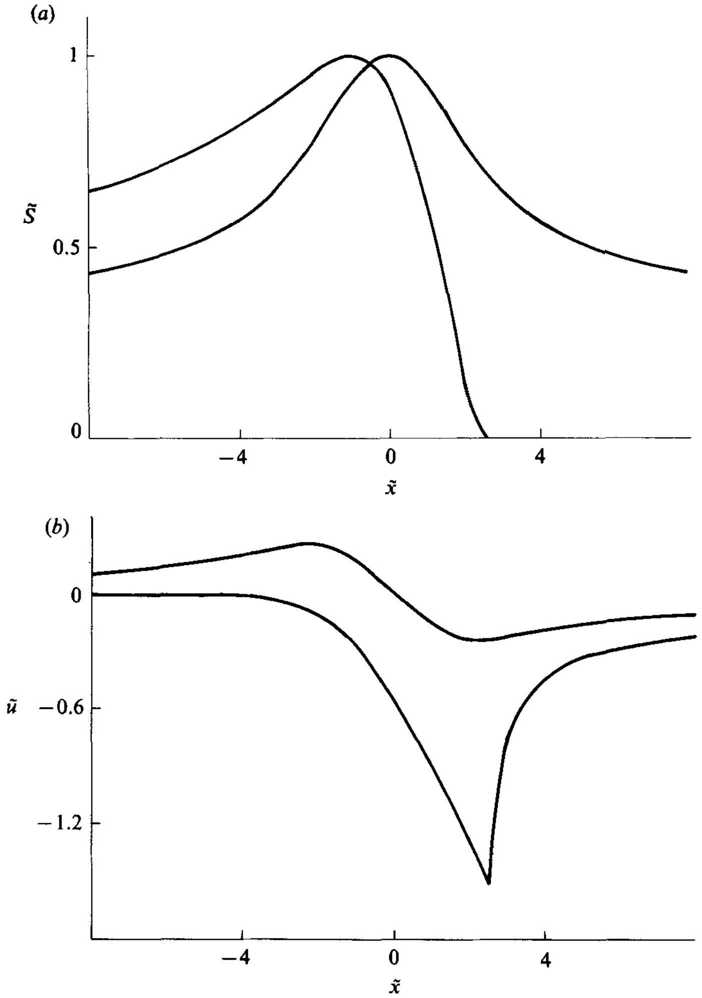

FIGURE 9(*a*). Subsonic eddy with $n = 0.6$: values of $\tilde{S}$ for symmetric eddy and for collapsed eddy. (*b*) Values of $\tilde{u}$ corresponding to $\tilde{S}$ in (*a*).

to those used previously. Derivatives in equations (3.3) were replaced by finite differences as in the discussion of supersonic breakdown in §4, and the Cauchy integral was treated as in §5.1. The limit solution for $\tilde{S}(\infty) = 0$ has, at its downstream end, a collapse similar to the downstream collapse of the symmetric $n = \frac{1}{2}$ eddy. As $\tilde{x} \rightarrow \tilde{x}_+ - 0$, both $\tilde{S}$ and $\tilde{S}'$ tend to zero although $\tilde{S}''$ is again infinite and the forms are

$$\tilde{S} \approx k(\tilde{x}_+ - \tilde{x})^{\frac{3}{2}}, \quad \tilde{u} = -n\tilde{x}_+ + \tfrac{1}{5}(4-3n)(\tilde{x}_+ - \tilde{x}), \tag{5.20}$$

while for $\tilde{x} > \tilde{x}_+$

$$\tilde{S} \equiv 0, \quad \tilde{u} \approx -n\tilde{x}_+ + (3k)^{\frac{1}{2}}(\tilde{x} - \tilde{x}_+)^{\frac{1}{4}}, \quad \tilde{p}' \approx -\tfrac{3}{4}k(\tilde{x} - \tilde{x}_+)^{-\frac{1}{2}}, \tag{5.21}$$

with $n = \frac{2}{3}$, which may be compared with (5.4), (5.5) respectively. The positive constant k, and the position $\tilde{x}_+$ of the cusp, follow from the numerical work and we obtain $\tilde{x}_+ \approx 2.24$, $k \approx 0.38$. The equations for the adjustment region again reduce to the single integral equation (5.12).

5.3. *The value* $n = 0.6$

For values of n such that $\frac{1}{2} \leqslant n < \frac{2}{3}$ the solution must be such that $\tilde{S}, \tilde{u} \rightarrow 0$ as $|\tilde{x}| \rightarrow \infty$. There appears to be one symmetric solution only (corresponding to $\tilde{S} \equiv 1$, $\tilde{u} \equiv 0$ when $n = \frac{2}{3}$ and to the symmetric collapsed eddy when $n = \frac{1}{2}$) with $\tilde{S}$ having its maximum value, of unity say, at the origin and $\tilde{u}$ an odd function of $\tilde{x}$. This maximum may be moved to the left until it reaches a critical position when the eddy collapses at some point $\tilde{x}_+ > 0$. This collapse is analogous to that experienced at $n = \frac{1}{2}$ and $n = \frac{2}{3}$, and indeed the results (5.20), (5.21) hold for general n. For $n = 0.6$, for example, we obtain $\tilde{x}_+ \approx 2.5$ and $k \approx 0.4$ and in figure 9(a, b) we show $\tilde{S}, \tilde{u}$ for the symmetric and for the collapsed eddy. The corresponding solution for upstream collapse may be deduced by applying (5.1) with $A = -1$.

6. Further discussion

The analysis presented here in §§3 and 5 suggests that an inviscid separated planar eddy, initially in a slowly evolving state of zero or uniform vorticity, can break down within a finite scaled time due to a fairly violent nonlinear interruption in the Kelvin–Helmholtz type of local interactive motion. This violent unsteady effect provides a link with the experimental observations of unsteady breakdown of separated flows noted in §1.

If this unsteady breakdown of the thin eddy does occur, its eruptive form, involving a focusing in the streamwise direction accompanied by stretching laterally and a faster temporal variation, leads on to a new stage where some extra physics comes into play (figure 10). The next new stage appears to be an Euler one where the typical slope of the eddy near its breakdown position becomes $O(1)$, the eddy length and width scales then both being $O(h^{\frac{1}{2}})$, for $n = \frac{1}{2}$, say, so that the typical pressure and velocity variations induced inside and outside are $O(1)$, while the timescale shortens considerably to $O(h^{\frac{1}{2}})$. The local motion outside and within the eddy there becomes controlled by the unsteady Euler equations, therefore, and a new vortex-sheet type of problem is posed for the determination of the unsteady eddy shape during that stage. This could well produce a rolling-up of the eddy boundary, or another kind of breakdown.

Moreover, it may be that other sections of the original thin eddy also go through the above process, leading then to multiple localized eruptions. These possible further implications seem to arise for a range of values of the index n introduced in the present study, including the favoured value $n = \frac{1}{2}$, incidentally. The value $\frac{1}{2}$ tends to be supported by integral constaints, that for undisturbed conditions upstream and downstream the streamwise integrals of $S(x, t)$ and $u(x, t)$ in (2.4) and (2.5) are conserved for all time, with (3.2) then providing $O(1)$ contributions to those integrals when $n = \frac{1}{2}$, and by the notion that the value $\frac{1}{2}$ gives the strongest accessible singularity. Again, the breakdown form is a nominally exact solution of the entire unsteady system of §2, with allowance made for the adjustment zones present when $n = \frac{1}{2}$ where smoothing is effected through small unsteady contributions or an effective slight change in the value of n.

The local eruptive process described theoretically above is similar in quality to some experimental observations on separating flow shown in Van Dyke's (1982) book and to the recent measurements of Dovgal *et al.* (1987), Kozlov (1987) and Mezaris *et al.* (1988) as well as to earlier experiments concerning the transition of separating flows; see also Gruber, Bestek & Fasel (1987) for a numerical simulation of the effect of

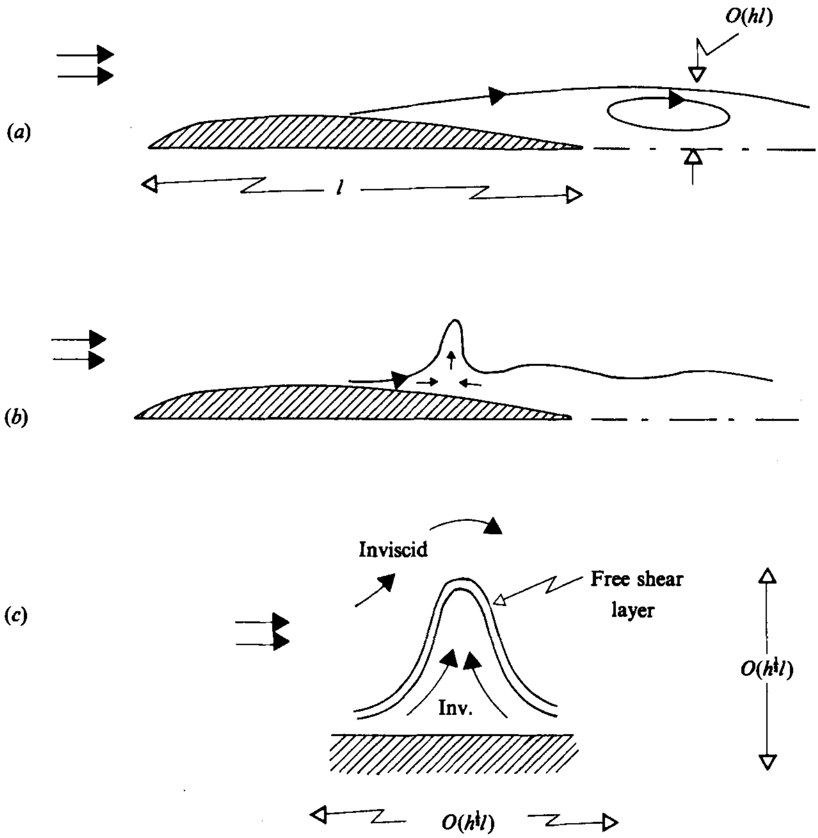

FIGURE 10. Schematic diagram of the unsteady nonlinear focusing and eruption predicted theoretically for an initially thin eddy. After the separated nonlinear eddy in (*a*) on a symmetric airfoil is disturbed, the eruption in (*b*) occurs and that leads on to the Euler stage of a vortex-sheet flow problem in (*c*).

a Tollmien–Schlichting wave on a separated flow. As Mezaris *et al.* comment, a fair degree of quantitative as well as qualitative agreement exists between their experimental results and the predictions, e.g. (1.1), of a theory related to the present study. That theory (Smith 1985*a*) is centred closer to the separation point but still beyond the triple-deck structure which controls the separation itself (and which Mezaris *et al.* seem to identify empirically in one of their figures), and the basic controlling equations in the incompressible/subsonic regime are

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}=-\frac{\partial P}{\partial X}, \tag{6.1a}$$

$$\frac{\partial S}{\partial T}+\frac{\partial}{\partial X}(US)=0, \tag{6.1b}$$

$$\frac{\partial}{\partial T}(S+A)+(S+A)\frac{\partial}{\partial X}(S+A)=-\frac{\partial P}{\partial X}, \tag{6.1c}$$

$$P(X,T)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial A}{\partial \xi}(\xi,T)\frac{\mathrm{d}\xi}{X-\xi}, \tag{6.1d}$$

in scaled terms, for the unknowns U, P, S, A as functions of X, T. Here U, P, S are essentially as in §2 with zero vorticity assumed but $-A$ stands for the boundary-layer displacement. A linearized version of (6.1) yields the earlier prediction (1.1), and there is a corresponding destabilization of the separated flow in the supersonic regime where (6.1d) is replaced by $P = -\partial A/\partial X$. Since the system (6.1) approaches that of §2 further downstream, however, where $A \approx -S$ in (6.1c), we expect sufficiently large disturbances in the local separated motion of (6.1) to erupt nonlinearly in much the same way as those discussed in the present work, thus forming the above link with the experimental findings.

A very similar finite-time breakdown is expected also to apply to the first stage of interaction in impulsively started flow past a circular cylinder. That interactive stage follows the classical boundary-layer breakdown of Van Dommelen (1981), and as noted by Elliott, Cowley & Smith (1983) its scaled governing equations are precisely (2.3) with (2.1). The boundary conditions are different, in that matching requires

$$u \propto y^{-2} \qquad \text{as } y \to 0+, \tag{6.2a}$$

$$u \propto (S-y)^{-2} \quad \text{as } y \to S-, \tag{6.2b}$$

and the flow solution starts from $t = -\infty$ in effect. Even so, an eruption of the present type seems most likely since (6.2a, b) can then play a secondary role, although the assumption of locally uniform vorticity with $n = \frac{1}{2}$ may well need to be modified. Given such an eruption (see also Smith 1985a; Henkes & Veldman 1987; Peridier, Smith & Walker 1988), the subsequent stage encountered locally on the cylinder is an Euler one again. Another application or extension of the present type of nonlinear breakdown with increasngly separated flow is to boundary-layer transition, via the unsteady triple-deck problem that governs the first nonlinear growth of Tollmien–Schlichting instabilities. Certain high-frequency/far-downstream properties are given by the nonlinear system (6.1) and others yield the classical breakdown and ensuing interactive structure where (6.2) comes into operation: Smith & Burggraf (1985). The problem involving (6.2) can also enter during a subsequent Euler stage of higher amplitudes. The entire unsteady triple-deck interaction in a subsonic or supersonic stream may also collapse nonlinearly with increasingly separated motion being induced; cf. the finite-time collapse of a complete interactive boundary layer described by Brotherton-Ratcliffe & Smith (1987).

Investigations of channel and pipe flows, wall-bounded jets and liquid-layer motions suggest that an analogous destabilization occurs when separated zones are present at high Reynolds numbers and these need to be pursued, as does the application to wind over water where there could be an interesting interplay between shallow-water properties and the pressure-displacement relation (2.1). In the present context of airfoil motion, non-symmetric wake features likewise seem well worth studying, for practical reasons at least, as do three-dimensional nonlinear effects and the influence of cross-flow. The three-dimensional subsonic version, for instance, of the simplified interaction (2.4), (2.5), (2.1) for zero or negligible vorticity is of the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}, \tag{6.3a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}, \tag{6.3b}$$

$$\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}(uS)+\frac{\partial}{\partial z}(wS)=0, \tag{6.3c}$$

$$p(x,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 S}{\partial \xi^2}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{[(x-\xi)^2+(z-\eta)^2]^{\frac{1}{2}}}. \tag{6.3d}$$

Like its planar-flow counterpart, (6.3) can become singular at a finite time with the $n=\frac{1}{2}$ option again but there may be other kinds of nonlinear breakdown possible of a more three-dimensional nature, just as three-dimensional linear modes are found to be more unstable than planar ones according to (6.3). In particular a highly three-dimensional version of (6.3) has u, w, p, S of order $\beta^l, l=\frac{5}{2}, 1, 2, 0$ in turn, with z, x, t of order $\beta^m, m=1, -\frac{1}{2}, 0$ respectively and β being small, in which case the main governing equations reduce to

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}, \tag{6.4a}$$

$$\frac{\partial S}{\partial t}+\frac{\partial}{\partial z}(wS)=0, \tag{6.4b}$$

$$p(x,z,t)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial \hat{S}}{\partial \eta}(x,\eta,t)\frac{\mathrm{d}\eta}{(z-\eta)}, \tag{6.4c}$$

with $\partial^2\hat{S}/\partial z^2 \equiv \partial^2 S/\partial x^2$. This form may reproduce the longitudinal vortex growth sometimes observed experimentally in separating flow. Throughout, viscous effects also need to be incorporated even if the predominantly inviscid breakdown associated with (6.2) tends to arise when viscous effects act initially within a passive unsteady boundary layer.

We note two final points: firstly a paper by Varley & Blythe (1983) was kindly pointed out to us by Professor J. D. A. Walker after the completion of our study, and it has a (rather loose) connection with our work; secondly the restriction $1 \gg h \gg R^{-\frac{1}{3}}$ applies for the basic equations of §2, in view of the mass flux in the eddy and in the separated shear layer.

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